Participation Factors for Nonlinear Systems in the Koopman Operator Framework

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Outline of Presentation

1. Motivation from the recent grid incident
2. Preliminaries
3. Nonlinear generalization --- the main result
4. Data-driven method
5. Summary and future work

Ref.)
K. Takamichi, Y. S., & M. Netto, *IEEE Transactions on Automatic Control* (conditionally accepted);
Preprint available at arXiv:2409.10105

PARTICIPATION FACTORS

A way of ***quantifying*** the impact of the i-th mode on the k-th state

The original definition	defined by means of the participation matrix $P = \{p_{ki}\} = \{u_{ki}v_{ki}\}$ (2)	Perez-Arriaga et al. (1982)
Use of high-order Taylor coefficients	$\sum_{j=1}^{n} u_{ij} z_{j0} e^{\lambda_j t} + \sum_{j=1}^{n} u_{ij} \left[\sum_{k=1}^{n} \sum_{l=1}^{n} h2^{j}_{kl} z_{k0} z_{l0} e^{(\lambda_k+\lambda_l)t} \right]$	Sanchez-Gasca et al. (2005)
Expectation -based	$p_{ki} := \operatorname*{avg}_{x^0 \in \mathcal{S}} \frac{(\ell^i x^0) r^i_k}{x^0_k}$	Hamzi & Abed (2020)
Expectation and Koopman-based	$p_{ij} := \mathbb{E}\left[\frac{\varphi_j(\boldsymbol{x}_0)\phi_{ij}}{\gamma_{i0}} \right] = \mathbb{E}\left[\frac{(\boldsymbol{\xi}_j^\top \boldsymbol{\gamma}_0)\phi_{ij}}{\gamma_{i0}} \right]$	Netto, Susuki, & Mili (2019)

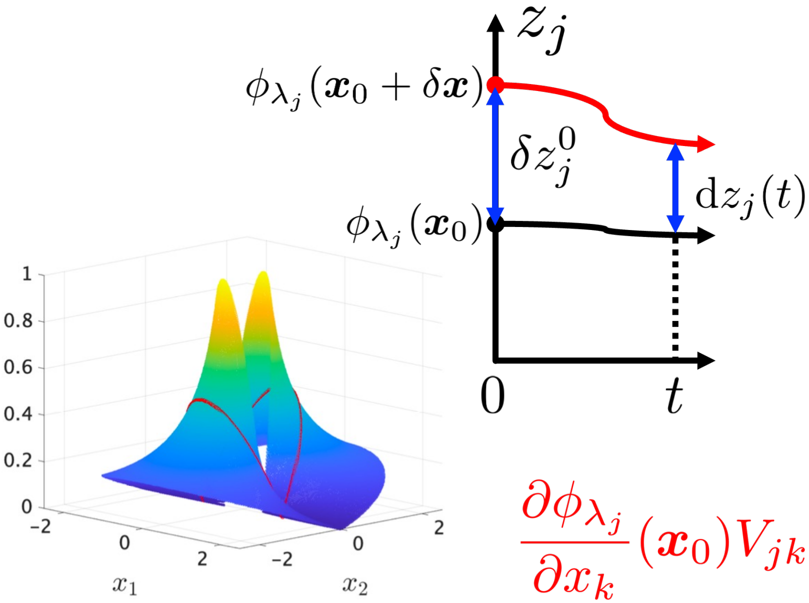
No system-theoretic foundation for nonlinear systems

Purpose & Contents

Theoretical foundation of Participation Factors for nonlinear autonomous dynamical systems

➢ Utilizing the Koopman operator framework, esp. Koopman mode decomposition and eigenfunctions

1. New formulation for linear systems
2. Nonlinear generalization --- the main result
3. Data-driven method



$$\frac{\partial \phi_{\lambda_j}}{\partial x_k}(\boldsymbol{x}_0)V_{jk}$$

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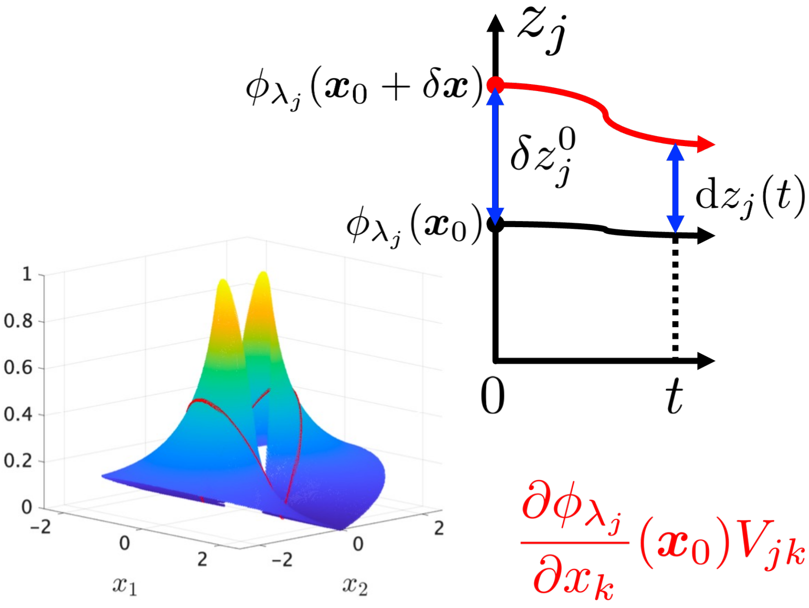
1. Motivation from the recent blackout
2. **Preliminaries**
   - **The original definition for linear systems**
   - **KMD and Koopman eigenfunctions**
3. Nonlinear generalization
4. Data-driven method
5. Summary and future work



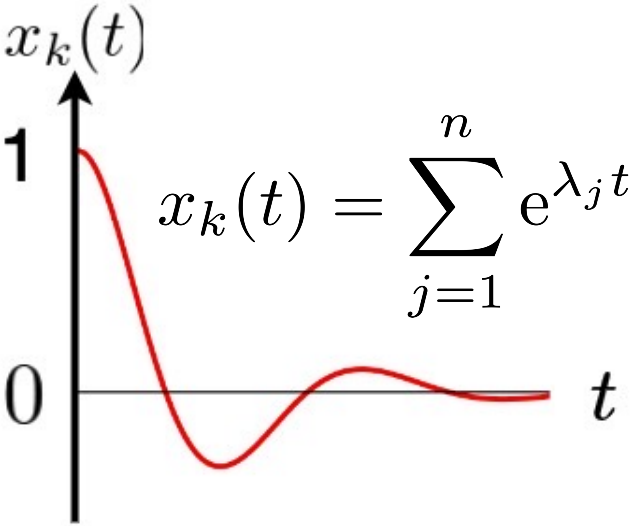
$$\frac{\partial\phi_{\lambda_j}}{\partial x_k}(\boldsymbol{x}_0)V_{jk}$$

The Original Definition

$$\dot{\boldsymbol{x}} = \mathsf{A}\boldsymbol{x}$$

$$\boldsymbol{u}_j^\top \mathsf{A} = \lambda_j \boldsymbol{u}_j^\top \qquad \mathsf{A}\boldsymbol{v}_j = \lambda_j \boldsymbol{v}_j$$

Ref.) Perez-Arriaga et al. (1982)



$$x_k(t) = \sum_{j=1}^{n} \mathrm{e}^{\lambda_j t} P_j^k$$

State's evolution:

j-th mode

$$\boldsymbol{x}(t) = \sum_{j=1}^{n} \mathrm{e}^{\lambda_j t} (\boldsymbol{u}_j^\top \boldsymbol{x}_0) \boldsymbol{v}_j$$

➡ $\boldsymbol{x}_0 = \boldsymbol{e}_k$

$$x_k(t) = \sum_{j=1}^{n} \mathrm{e}^{\lambda_j t} \boxed{u_{jk} v_{jk}}$$

k-th

P_j^k Mode-in-State PF

Mode's evolution:

State-in-Mode PF

$$z_j(t) = \boldsymbol{u}_j^\top \sum_{i=1}^{n} \mathrm{e}^{\lambda_i t} (\boldsymbol{u}_i^\top \boldsymbol{x}_0) \boldsymbol{v}_i = \mathrm{e}^{\lambda_j t} \sum_{k=1}^{n} \boxed{u_{jk} v_{jk}}$$

j-th

$$z_j(0) = \boldsymbol{u}_j^\top \boldsymbol{v}_j = 1$$

The Key Table

	LINEAR	NONLINEAR
Model	$\dot{\boldsymbol{x}} = \mathsf{A}\boldsymbol{x}$	$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$
Mode Expansion	$\sum_{j=1}^{n} \mathrm{e}^{\lambda_j t} \boldsymbol{u}_j^\top \boldsymbol{x}_0 \boldsymbol{v}_j$	?
Participation Factors	$u_{jk} v_{jk}$ $\boldsymbol{u}_j^\top \mathsf{A} = \lambda_j \boldsymbol{u}_j^\top$ $\mathsf{A}\boldsymbol{v}_j = \lambda_j \boldsymbol{v}_j$	?

Refs.) Takamichi et al. (2022,2024)

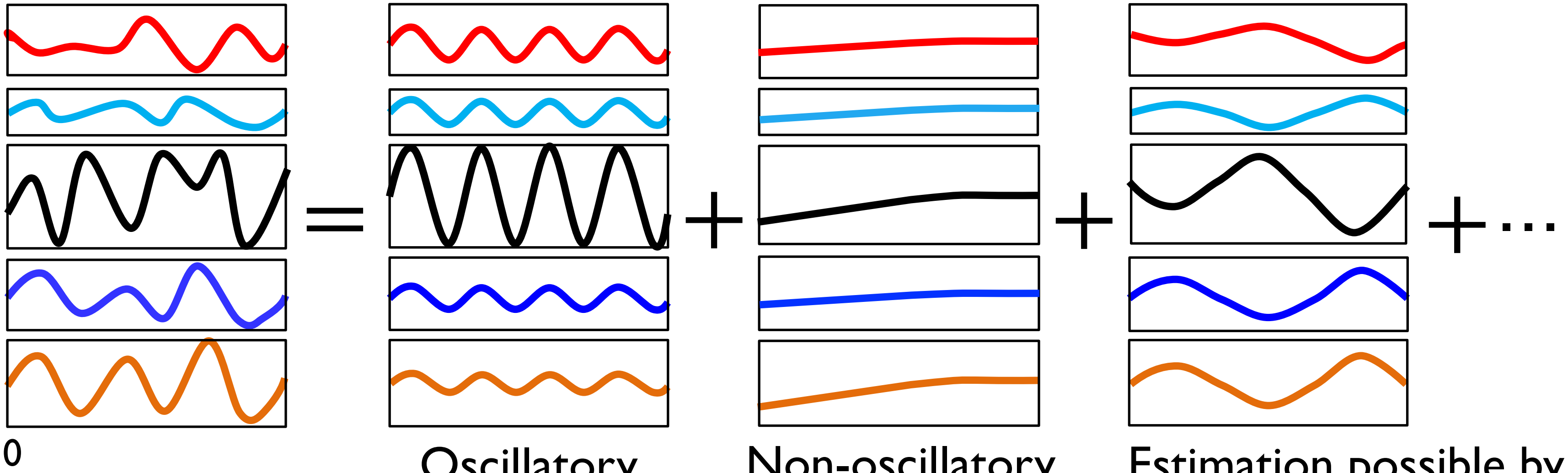
Koopman Mode Decomposition

Refs.) Mezic (2005); Rowley et al. (2009)

State's evolution:

$$\boldsymbol{x}(t) = \sum_{j=1}^{\infty} \mathrm{e}^{\lambda_j t} \phi_{\lambda_j}(\boldsymbol{x}(0)) \boldsymbol{V}_j$$

Decomposition of nonlinear dynamics based on ***Koopman eigenvalues, eigenfunctions,*** & ***modes***



Nonlinear evolution

Oscillatory mode (single-freq.)

Non-oscillatory mode

Estimation possible by DMD (Dynamic Mode Decomposition)

Properties of Koopman Eigenvalues

THM: For ***analytic*** vector fields w/ ***globally stable equilibrium points*** & ***analytic*** observables, their Koopman spectra consist of the ***point*** only.

$$U^t\phi_{\lambda_j}(\boldsymbol{x}) = \mathrm{e}^{\lambda_j t}\phi_{\lambda_j}(\boldsymbol{x})$$

Refs.) Mauroy et al. (2013)
Mezic (2020)

THM: For ***analytic*** vector fields w/ ***globally stable limit cycles*** & ***analytic*** observables, their Koopman spectra consist of the ***point*** only.

Refs.) Mauroy & Mezic (2018); Mezic (2020)

THM: The associated Koopman eigenfunctions are ***smooth***.

Refs.) Mauroy et al. (2013); Mauroy & Mezic (2018)
Mezic (2020); Kvalheim & Revzen (2021)

The Key Table

	LINEAR	NONLINEAR
Model	$\dot{\boldsymbol{x}} = \mathsf{A}\boldsymbol{x}$	$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$
Mode Expansion	$\sum_{j=1}^{n} \mathrm{e}^{\lambda_j t} \boldsymbol{u}_j^\top \boldsymbol{x}_0 \boldsymbol{v}_j$	$\sum_{j=1}^{\infty} \mathrm{e}^{\lambda_j t} \phi_{\lambda_j}(\boldsymbol{x}_0) \boldsymbol{V}_j$
Participation Factors	$u_{jk} v_{jk}$ $\boldsymbol{u}_j^\top \mathsf{A} = \lambda_j \boldsymbol{u}_j^\top$ $\mathsf{A}\boldsymbol{v}_j = \lambda_j \boldsymbol{v}_j$	?

Refs.) Takamichi et al. (2022,2024)

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2. Preliminaries
3. **Nonlinear generalization --- the main result**
    - **New formulation**
    - **Koopman-based result**
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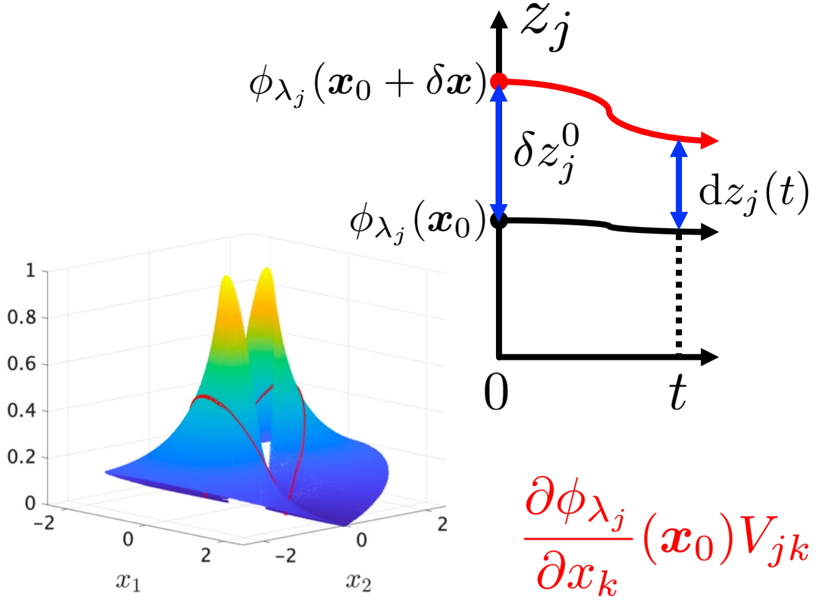


$$\frac{\partial \phi_{\lambda_j}}{\partial x_k}(\boldsymbol{x}_0) V_{jk}$$

The Original Definition

$\dot{\boldsymbol{x}} = \mathsf{A}\boldsymbol{x}$

Ref.) Perez-Arriaga et al. (1982)

$\boldsymbol{u}_j^\top \mathsf{A} = \lambda_j \boldsymbol{u}_j^\top \qquad \mathsf{A}\boldsymbol{v}_j = \lambda_j \boldsymbol{v}_j$

State's evolution:

j-th mode

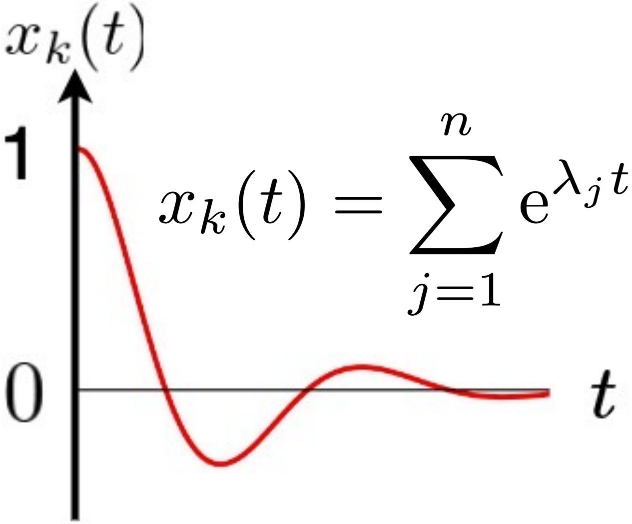
$$\boldsymbol{x}(t) = \sum_{j=1}^{n} \mathrm{e}^{\lambda_j t} \boxed{(\boldsymbol{u}_j^\top \boldsymbol{x}_0)} \boldsymbol{v}_j$$

$\boldsymbol{x}_0 = \boldsymbol{e}_k$

$$\boxed{x_k(t)} = \sum_{j=1}^{n} \mathrm{e}^{\lambda_j t} \boxed{u_{jk} v_{jk}}$$

k-th

P_j^k Mode-in-State PF



$$x_k(t) = \sum_{j=1}^{n} \mathrm{e}^{\lambda_j t} P_j^k$$

The initialization works only for linear systems. Another idea?

$\boldsymbol{x}_0 = \boldsymbol{v}_j$

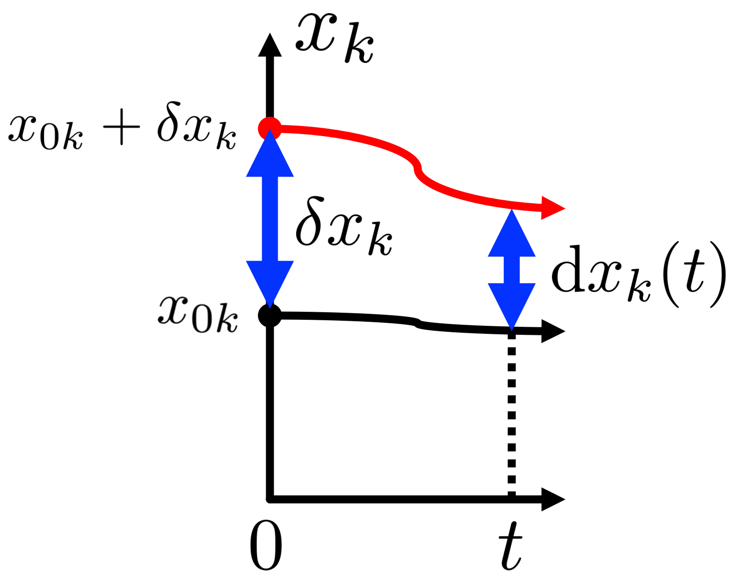
$z_j(0) = \boldsymbol{u}_j^\top \boldsymbol{v}_j = 1$

Variational Formulation --- Mode-in-State

Ref.) Takamichi et al. (2024)

LEMMA: For ***linear systems***,

$$\overset{\text{Variation}}{\boxed{\mathrm{d}x_k(t;\delta\boldsymbol{x})}} = \sum_{j=1}^{n} \mathrm{e}^{\lambda_j t} \overset{\text{PF}}{\underset{u_{jk}v_{jk}}{P_j^k}} \boxed{\delta x_k} + \sum_{j=1}^{n}\sum_{\substack{\ell=1\\ \ell\neq k}}^{n} \mathrm{e}^{\lambda_j t} \overset{\text{Generalized Participation}}{\underset{u_{j\ell}v_{jk}}{P_j^{k(\ell)}}} \delta x_\ell$$



- PFs quantifying the contributions of the modes ***to the variational dynamics of the states***
- PFs derived w/o ***initialization***
- Direct to the ***nonlinear generalization***

Nonlinear Generalization --- Mode-in-State

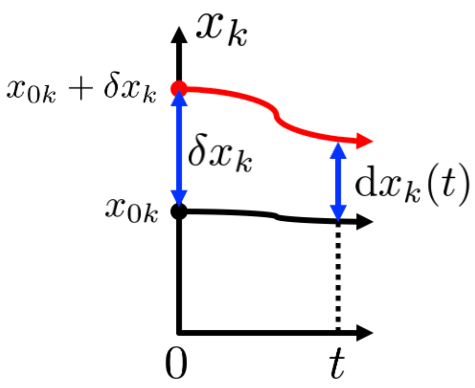
THM: For ***nonlinear systems***,

$$\boxed{\mathrm{d}x_k(t;\boldsymbol{x}_0,\delta\boldsymbol{x})} \approx \sum_{j=1}^{n} \mathrm{e}^{\lambda_j t} \overset{\frac{\partial \phi_{\lambda_j}}{\partial x_k}(\boldsymbol{x}_0)V_{jk}}{\underbrace{P_j^k(\boldsymbol{x}_0)}_{\text{PF}}}\boxed{\delta x_k} + \sum_{\substack{\ell=1\\ \ell\neq k}}^{n}\sum_{j=1}^{n} \mathrm{e}^{\lambda_j t} \overset{\frac{\partial \phi_{\lambda_j}}{\partial x_\ell}(\boldsymbol{x}_0)V_{jk}}{\underbrace{P_j^{k(\ell)}(\boldsymbol{x}_0)}_{\text{Generalized Participation}}}\delta x_\ell$$

$$+ \sum_{\substack{j_1,\ldots,j_n\in\mathbb{N}_0\\ j_1+\cdots+j_n>1}}^{\infty} \mathrm{e}^{(j_1\lambda_1+\cdots+j_n\lambda_n)t} P_{\langle j_1\cdots j_n\rangle}^{k}(\boldsymbol{x}_0)\delta x_k$$

High-order

$$+ \sum_{\substack{\ell=1\\ \ell\neq k}}^{n}\sum_{\substack{j_1,\ldots,j_n\in\mathbb{N}_0\\ j_1+\cdots+j_n>1}}^{\infty} \mathrm{e}^{(j_1\lambda_1+\cdots+j_n\lambda_n)t} P_{\langle j_1\cdots j_n\rangle}^{k(\ell)}(\boldsymbol{x}_0)\delta x_\ell$$



- PFs defined with ***partial derivatives of Koopman eigenfunctions and modes***
- PFs ***depending on*** the initial state

- PFs ***defined globally***
- Linear case as a Special Case

$$\phi_{\lambda_j}(\boldsymbol{x}) = \boldsymbol{u}_j^\top \boldsymbol{x} \quad \boldsymbol{V}_j = \boldsymbol{v}_j$$

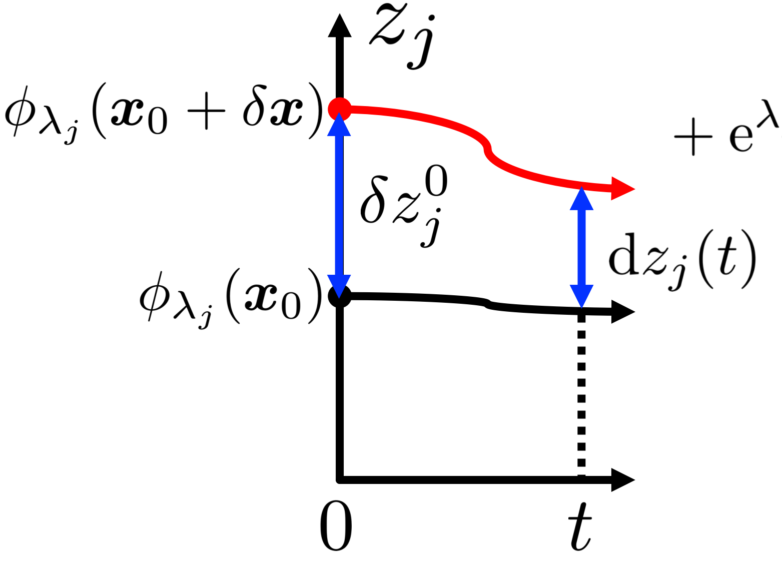
Nonlinear Generalization --- State-in-Mode

LEMMA: Mode variable $z_j = \phi_{\lambda_j}(\boldsymbol{x})$

THM: For ***nonlinear systems***,

$$\boxed{\mathrm{d}z_j(t;\boldsymbol{x}_0,\delta\boldsymbol{x})} \approx \mathrm{e}^{\lambda_j t}\sum_{k=1}^{n} \overset{\frac{\partial\phi_{\lambda_j}}{\partial x_k}(\boldsymbol{x}_0)V_{jk}}{\underset{\text{PF}}{P_j^k(\boldsymbol{x}_0)}}\boxed{\delta z_j^0} + \mathrm{e}^{\lambda_j t}\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{k=1}^{n} \overset{\frac{\partial\phi_{\lambda_j}}{\partial x_k}(\boldsymbol{x}_0)V_{ik}}{P_{i(j)}^k(\boldsymbol{x}_0)}\delta z_i^0$$

$$+\,\mathrm{e}^{\lambda_j t}\sum_{k=1}^{n}\sum_{\substack{j_1,\ldots,j_n\in\mathbb{N}_0\\ j_1+\cdots+j_n>1}}^{\infty} \underset{\text{high-order}}{P_{\langle j_1\cdots j_n\rangle(j)}^k}(\boldsymbol{x}_0)\delta z_{\langle j_1\cdots j_n\rangle}^0$$



- PFs defined with partial derivatives of Koopman eigenfunctions and modes, depending on the initial state, and defined globally

The Key Table

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Mode Expansion	$\sum_{j=1}^{n} \mathrm{e}^{\lambda_j t} \boldsymbol{u}_j^\top \boldsymbol{x}_0 \boldsymbol{v}_j$	$\sum_{j=1}^{\infty} \mathrm{e}^{\lambda_j t} \phi_{\lambda_j}(\boldsymbol{x}_0) \boldsymbol{V}_j$
Participation Factors	$u_{jk} v_{jk}$ $\boldsymbol{u}_j^\top \mathsf{A} = \lambda_j \boldsymbol{u}_j^\top$ $\mathsf{A}\boldsymbol{v}_j = \lambda_j \boldsymbol{v}_j$	$\frac{\partial \phi_{\lambda_j}}{\partial x_k}(\boldsymbol{x}_0) V_{jk}$

Valid for Systems with Stable Equilibriums and Limit Cycles

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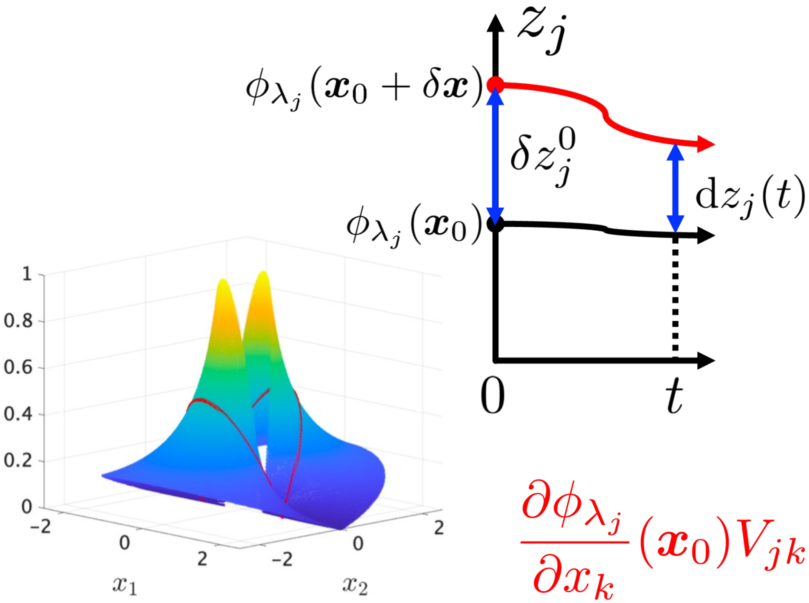


$$\frac{\partial \phi_{\lambda_j}}{\partial x_k}(\boldsymbol{x}_0) V_{jk}$$

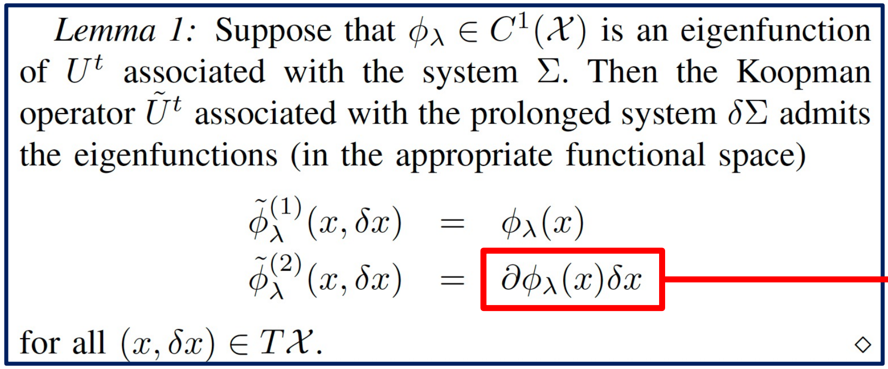
Data-Driven Method

- By definition, we need to estimate the Koopman eigenfunction (esp., ***its partial derivative***).

- ***No need!*** - by using the ***prolonged formulation***, with the DMD algorithm

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{x}(t))$$

$$\delta\dot{\boldsymbol{x}}(t) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}(\boldsymbol{x}(t))\delta\boldsymbol{x}(t)$$



Lemma 1: Suppose that $\phi_\lambda \in C^1(\mathcal{X})$ is an eigenfunction of U^t associated with the system Σ. Then the Koopman operator $\tilde{U}^t$ associated with the prolonged system $\delta\Sigma$ admits the eigenfunctions (in the appropriate functional space)

$$\begin{aligned} \tilde{\phi}_\lambda^{(1)}(x, \delta x) &= \phi_\lambda(x) \\ \tilde{\phi}_\lambda^{(2)}(x, \delta x) &= \boxed{\partial\phi_\lambda(x)\delta x} \end{aligned}$$

for all $(x, \delta x) \in T\mathcal{X}$. ◇

from Mauroy, Forni, & Sepulchre (2015)

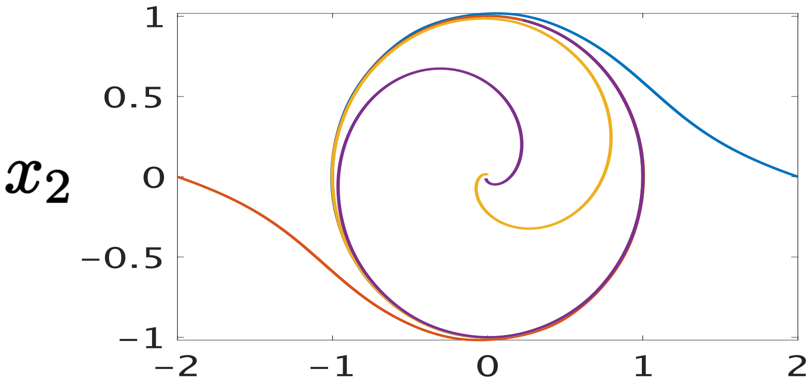
$$\frac{\partial \phi_{\lambda_j}}{\partial x_k}(\boldsymbol{x}_0)V_{jk}$$

Estimated as DMD modes directly from numerical solutions

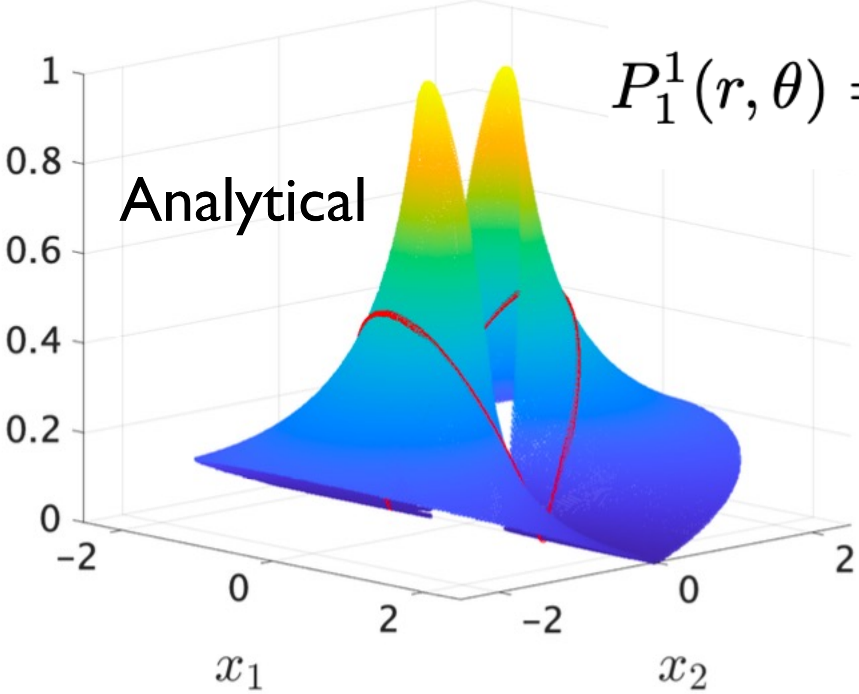
Numerical Example for Limit-Cycle System

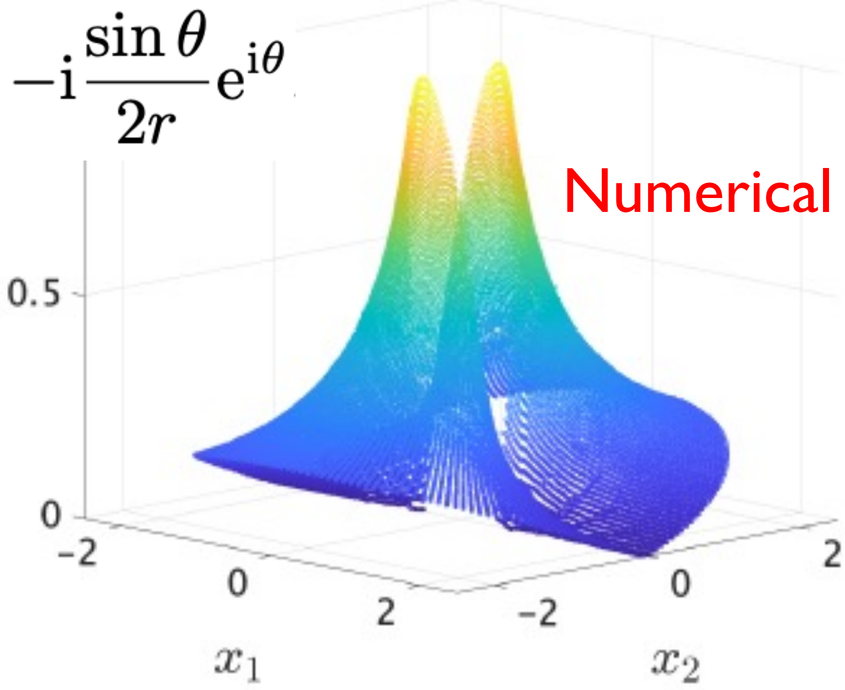
$$\dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2)$$

$$\Rightarrow \left.\begin{aligned} \dot{r} &= r - r^3 \\ \dot{\theta} &= 1 \end{aligned}\right\}$$



$$P_1^1(r, \theta) = -\mathrm{i}\frac{\sin\theta}{2r}\mathrm{e}^{\mathrm{i}\theta}$$





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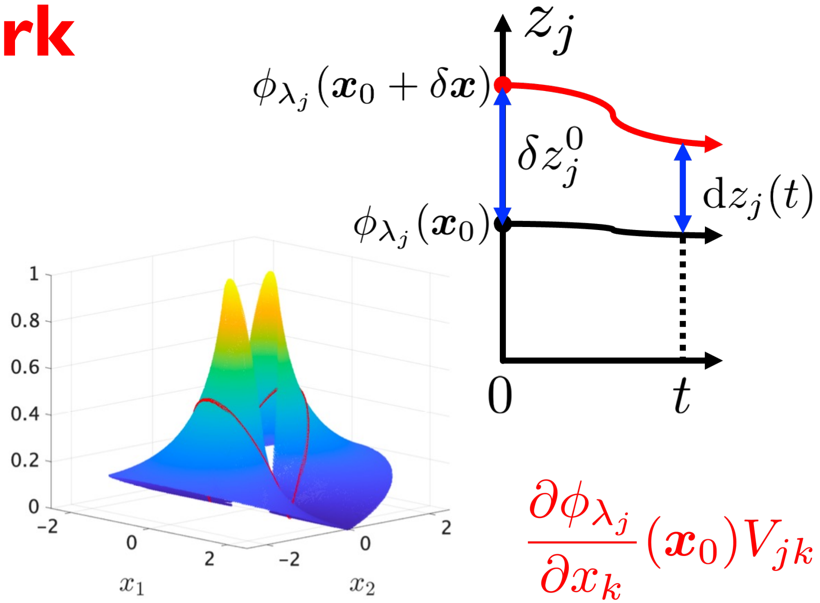
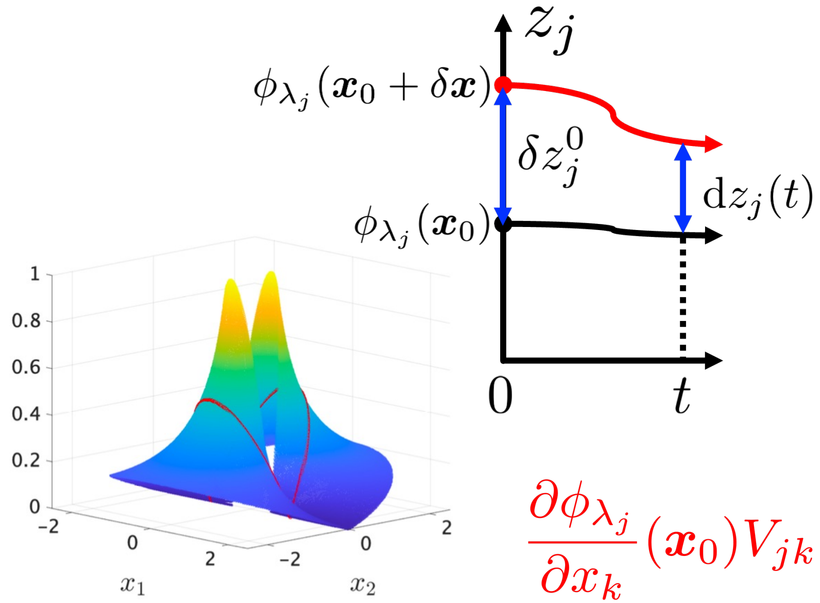


$$\frac{\partial \phi_{\lambda_j}}{\partial x_k}(\boldsymbol{x}_0)V_{jk}$$

Summary --- The Three Messages

Theoretical foundation of Participation Factors for nonlinear autonomous dynamical systems

1. The ***variational formulation*** enables the unified definition of participation factors.
2. ***Partial derivative of Koopman eigenfunctions*** quantifies the nonlinear participation.
3. Data-driven method is possible ***w/o estimating the eigenfuncion***.



$$\frac{\partial \phi_{\lambda_j}}{\partial x_k}(\boldsymbol{x}_0)V_{jk}$$

Future Directions

- Extend it to a ***mode-in-observable*** PF
- Find a connection with the ***Lyapunov exponent***
- Apply it to real problems, such as the ***power grid***
- Combine it with the ***data assimilation technique*** in weather prediction (and control)

Ref.)
K. Takamichi, Y. S., & M. Netto, *IEEE Transactions on Automatic Control* (conditionally accepted);
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